Katedra matematiky Fakulty jaderné a fyzikálně inženýrské ČVUT v Praze								CELKEM
Jméno a příjmení	Cvičící	1	2	3	4	5	6	

Zápočtová práce č. 3 z předmětu 01ANB3 – verze A

8. ledna 2024   13:00 — 14:40

Může se hodit: $\displaystyle\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$

Jex

1 (7 bodů)

Sestavte první a druhý totální diferenciál v bodě $(x_0, y_0) = (2, 0)$ pro funkci

$$f(x,y) = \ln\left(\sqrt{x^2+y^2} + y\right).$$

Jex

2 (6 bodů)

Na $\mathbf{R}^2$ je zavedena metrika předpisem

$$\varrho(\vec{x},\vec{y}) = |x_1 - y_1| + \lceil\, |x_2 - y_2|\, \rceil.$$

Nechť je dána množina $A = \{\vec{x} \in \mathbf{R}^2 : |x_1| + |x_2| \leqslant 1\}$. Kolik prvků (a jaké) má množina $A \setminus A'$? Symbol A' reprezentuje derivaci množiny A.

Vrbálek

3 (7 bodů)

Do vlastního obrázku vykreslete množinu, vzhledem k níž je funkce

$$h(x,y) = \begin{cases} \frac{4x(y-1)^3}{x^2+3(y-1)^6} & \leftrightsquigarrow (x,y) \neq (0,1) \\ 1 & \leftrightsquigarrow (x,y) = (0,1) \end{cases}$$

spojitá v bodě $\vec{a} = (0, 1)$. V obrázku nezapomeňte popsat osy a naznačit měřítko.

4 (7 bodů) Kovář

Hessovy matice funkcí $f(x,y)$ a $g(x,y)$ v bodě $\vec{a}$ jsou tvaru

$$\mathbb{F} = \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}, \quad \mathbb{G} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

a jejich gradienty jsou $\operatorname{grad} f(\vec{a}) = \operatorname{grad} g(\vec{a}) = (-3, 3)$. Navíc $f(\vec{a}) = 2$ a $g(\vec{a}) = -4$. Jaký tvar má Hessova matice funkce $h(x,y) = f(x,y) \cdot g(x,y)$? Co lze vypozorovat ze tvaru matice $\mathbb{H}$?

5 (7 bodů) Kovanda

Na Hilbertově prostoru $\mathscr{H}$ jistých funkcí definovaných na intervalu $\langle 0, +\infty)$ je zadán skalární součin

$$\langle f|g\rangle := \int_0^\infty f(x)g(x)\,e^{-2x}\,dx.$$

Vypočtěte úhel, který v $\mathscr{H}$ svírají funkce $f(x) = x^2$ a $g(x) = x$.

6 (6 bodů) Klika

Vypočítejte směrovou derivaci funkce $f(x,y,z) = 10\,\mathrm{arctg}(xy + y^2) + yz^2$ ve směru $\vec{s} = (2, -1, -2)$ v bodě $(x_0, y_0, z_0) = (2, 1, -1)$.

In[53]:= `f = Log[y + Sqrt[y^2 + x^2]]`

Out[53]= $\text{Log}\left[y+\sqrt{x^2+y^2}\right]$

In[54]:= `A = D[f, {x, 1}] // FullSimplify`

Out[54]= $\frac{x}{x^2+y\left(y+\sqrt{x^2+y^2}\right)}$

In[55]:= `B = D[f, {y, 1}] // FullSimplify`

Out[55]= $\frac{1}{\sqrt{x^2+y^2}}$

In[56]:= `AA = D[A, {x, 1}] // FullSimplify`

Out[56]= $\frac{-x^2+y^2+\frac{y^3}{\sqrt{x^2+y^2}}}{\left(x^2+y\left(y+\sqrt{x^2+y^2}\right)\right)^2} = \frac{\partial^2 f}{\partial x^2}$

In[57]:= `BB = D[B, {y, 1}] // FullSimplify`

Out[57]= $-\frac{y}{\left(x^2+y^2\right)^{3/2}} = \frac{\partial^2 f}{\partial y^2}$

In[58]:= `CC = D[B, {x, 1}] // FullSimplify`

Out[58]= $-\frac{x}{\left(x^2+y^2\right)^{3/2}} = \frac{\partial^2 f}{\partial y \partial x}$

In[60]:= `{f, A, B, AA, BB, CC} /. {x → 2, y → 0} // FullSimplify`

Out[60]= $\left\{\text{Log}[2], \frac{1}{2}, \frac{1}{2}, -\frac{1}{4}, 0, -\frac{1}{4}\right\}$

$$\frac{\partial f}{\partial x}(\vec{a}) = \frac{1}{2} \checkmark$$

$$\frac{\partial f}{\partial y}(\vec{a}) = \frac{1}{2} \checkmark$$

$$\frac{\partial^2 f}{\partial x^2}(\vec{a}) = -\frac{1}{4} \checkmark$$

$$\frac{\partial^2 f}{\partial y^2}(\vec{a}) = 0 \checkmark$$

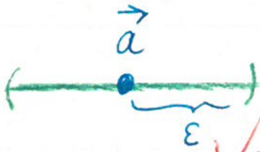
$$\frac{\partial^2 f}{\partial y \partial x}(\vec{a}) = \frac{\partial^2 f}{\partial x \partial y}(\vec{a}) = -\frac{1}{4} \checkmark$$

$$df_{\vec{a}}(\vec{h}) = \frac{1}{2} h_1 + \frac{1}{2} h_2 \checkmark$$

$$d^2 f_{\vec{a}}(\vec{h}) = -\frac{1}{4} h_1^2 - 2\,\frac{1}{4} h_1 h_2 = -\frac{1}{4} h_1^2 - \frac{1}{2} h_1 h_2 \checkmark$$

$$\mathbb{H}_{\vec{a}} = \begin{pmatrix} -1/4 & -1/4 \\ -1/4 & 0 \end{pmatrix}$$

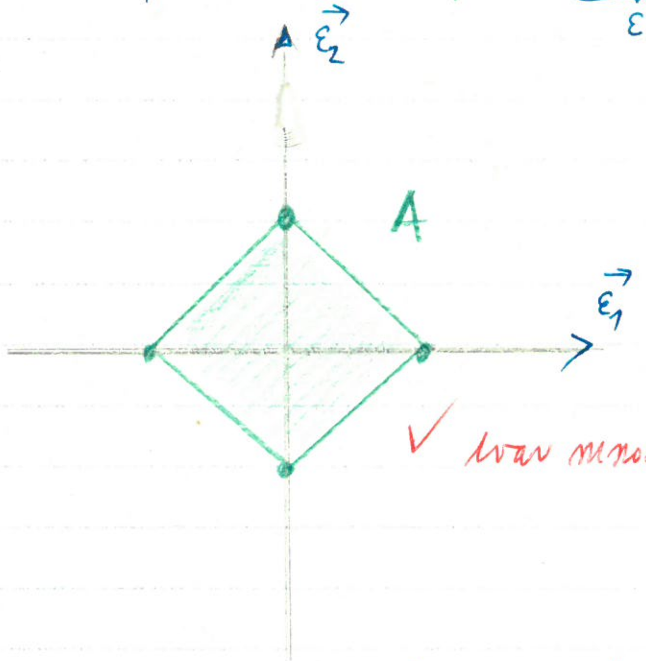
Malé okolí bodu $\vec{a}$ (tj. pro $\varepsilon < 1$) je tvaru:



6b

✓ tvar okolí

Množina A:



✓ tvar množiny

$A \setminus A'$ je vlastně množina izolovaných bodů množiny A ✓ pochopení úlohy

tj. hledáme izolované body. Pro ně musí platit, že existuje $\varepsilon > 0$ tak, že $A \cup U_\varepsilon(\vec{b}) = \{\vec{b}\}$

Takové body jsou jen dva: $\vec{b} = (0,1)$ & $\vec{c} = (0,-1)$

Závěr:

$$A \setminus A' = \{(0,1); (0,-1)\}$$

✓✓✓

7b

$$h(x,y) = \begin{cases} \dfrac{4x(y-1)^3}{x^2+3(y-1)^6} & (x,y) \neq (0,1) \\ 1 & (x,y) = (0,1) \end{cases}$$

Množina musí procházet bodem $\vec{a} = (0,1)$ a musí pro ni platit:

$$\lim_{\substack{\vec{x} \to \vec{a} \\ \vec{x} \in M}} h(x,y) \overset{!}{=} 1$$

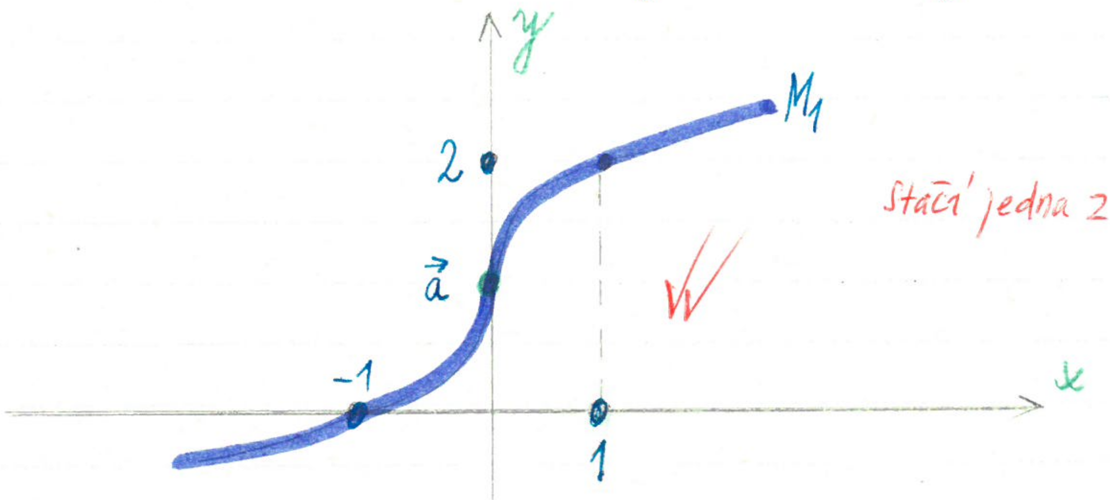
Volíme systém množin $M_\alpha = \{(x,y) \in \mathbb{R}^2 : x = \alpha(y-1)^3\}$

$$\lim_{\substack{(x,y)\to(0,1) \\ (x,y)\in M_\alpha}} h(x,y) = \lim_{y\to 1} \frac{4\alpha(y-1)^3(y-1)^3}{\alpha^2(y-1)^6+3(y-1)^6} = \frac{4\alpha}{\alpha^2+3} \overset{!}{=} 1$$

$$\alpha^2 - 4\alpha + 3 = 0$$
$$(\alpha-1)(\alpha-3) = 0$$

$\alpha_1 = 1$ & $\alpha_2 = 3$

Máme dvě řešení:
$$M_1 = \{(x,y) \in \mathbb{R}^2 : x = (y-1)^3\}$$
$$M_2 = \{(x,y) \in \mathbb{R}^2 : x = 3(y-1)^3\}$$



Stačí jedna z množin

$\mathbb{F} = \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}$ $\quad \mathbb{G} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad \operatorname{grad} f(\vec{a}) = \operatorname{grad} g(\vec{a}) = (-3, 3)$

$f(\vec{a}) = 2 \qquad g(\vec{a}) = -4$

$h(x,y) = f(x,y) \cdot g(x,y)$

$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial x} \cdot g + f \cdot \frac{\partial g}{\partial x} \qquad \frac{\partial h}{\partial y} = \frac{\partial f}{\partial y} \cdot g + f \cdot \frac{\partial g}{\partial y}$$

$$\frac{\partial^2 h}{\partial x^2} = \underbrace{\frac{\partial^2 f}{\partial x^2}}_{1} \cdot \underbrace{g}_{-4} + \underbrace{\frac{\partial f}{\partial x}}_{-3} \cdot \underbrace{\frac{\partial g}{\partial x}}_{-3} \cdot 2 + \underbrace{f \cdot \frac{\partial^2 g}{\partial x^2}}_{2 \cdot 2} \Rightarrow \frac{\partial^2 h}{\partial x^2}(\vec{a}) = -4 + 18 + 4 = +18$$

$$\frac{\partial^2 h}{\partial y^2} = \underbrace{\frac{\partial^2 f}{\partial y^2}}_{1} \cdot \underbrace{g}_{-4} + 2 \cdot \underbrace{\frac{\partial f}{\partial y} \cdot \frac{\partial g}{\partial y}}_{3 \cdot 3} + \underbrace{f \cdot \frac{\partial^2 g}{\partial y^2}}_{2 \cdot 2} \Rightarrow \frac{\partial^2 h}{\partial y^2}(\vec{a}) = -4 + 18 + 4 = 18$$

$$\frac{\partial^2 h}{\partial x \partial y} = \underbrace{\frac{\partial^2 f}{\partial x \partial y} \cdot g}_{3 \cdot (-4)} + \underbrace{\frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial y}}_{-3 \cdot 3} + \underbrace{\frac{\partial f}{\partial y} \cdot \frac{\partial g}{\partial x}}_{3 \cdot (-3)} + \underbrace{f \cdot \frac{\partial^2 g}{\partial x \partial y}}_{2 \cdot 1} \Rightarrow \frac{\partial^2 h}{\partial x \partial y}(\vec{a}) = -12 - 18 + 2 = -28$$

$$\frac{\partial^2 h}{\partial y \partial x} = \underbrace{\frac{\partial^2 f}{\partial y \partial x} \cdot g}_{(-1) \cdot (-4)} + \underbrace{\frac{\partial f}{\partial y} \cdot \frac{\partial g}{\partial x}}_{3 \cdot (-3)} + \underbrace{\frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial y}}_{-3 \cdot 3} + \underbrace{f \cdot \frac{\partial^2 g}{\partial y \partial x}}_{2 \cdot 1} \Rightarrow \frac{\partial^2 h}{\partial y \partial x}(\vec{a}) = +4 - 18 + 2 = -12$$

$$H_{\vec{a}} = \begin{pmatrix} 18 & -28 \\ -12 & 18 \end{pmatrix}$$

$$\Rightarrow \underbrace{h(x,y) \notin C_2(U_\delta(\vec{a})) \text{ na žádném } U_\delta(\vec{a})}$$

! Nelze míchat pořadí derivací !

7b

$$\langle a|b\rangle = \int_0^{+\infty} a(x)\cdot b(x)\, e^{-2x}\, dx$$

```
In[62]:= f = x^2
         g = x
         a = 2
Out[62]= x^2
Out[63]= x
Out[64]= 2
In[65]:= AA = Integrate[f*g*Exp[-a*x], {x, 0, Infinity}]
Out[65]= 3/8
In[66]:= BB = Integrate[f*f*Exp[-a*x], {x, 0, Infinity}]
Out[66]= 3/4
In[67]:= CC = Integrate[g*g*Exp[-a*x], {x, 0, Infinity}]
Out[67]= 1/4
In[68]:= fi = Assuming[Re[a] > 0, AA / Sqrt[BB*CC]]
Out[68]= Sqrt[3]/2
In[69]:= uhel = ArcCos[fi]
Out[69]= π/6
```

$$\int_0^{\infty} x^n e^{-ax}\, dx = \frac{n!}{a^{n+1}}$$

$f(x) = x^2$ & $g(x) = x$

$$\langle f|g\rangle = \int_0^{\infty} x^2\cdot x\cdot e^{-2x}\, dx = \int_0^{\infty} x^3 e^{-2x}\, dx = \frac{3}{8}$$

$$\langle f|f\rangle = \int_0^{\infty} x^2\cdot x^2 e^{-2x}\, dx = \int_0^{\infty} x^4 e^{-2x}\, dx = \frac{4\cdot 3\cdot 2}{4\cdot 4\cdot 2} = \frac{3}{4}$$

$$\langle g|g\rangle = \int_0^{\infty} x\cdot x\cdot e^{-2x}\, dx = \int_0^{\infty} x^2 e^{-2x}\, dx = \frac{2}{2^3} = \frac{1}{4}$$

$$s = \frac{|\langle f|g\rangle|}{\sqrt{\langle f|f\rangle\cdot\langle g|g\rangle}} = \frac{\frac{3}{8}}{\frac{\sqrt{3}}{2}\cdot\frac{1}{2}} = \frac{3}{\sqrt{3}\cdot 2} = \frac{\sqrt{3}}{2}$$

$$\varphi = \arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

!

6b

```
In[10]:= f = 10 * ArcTan[x * y + y^2] + y z^2
```

Out[10]= $y z^2 + 10\,\mathrm{ArcTan}\left[x y + y^2\right]$

```
In[11]:= AA = D[f, {x, 1}]
```

Out[11]= $\dfrac{10\,y}{1 + \left(x y + y^2\right)^2} = \dfrac{\partial f}{\partial x}$ ✓

```
In[12]:= BB = D[f, {y, 1}]
```

Out[12]= $\dfrac{10\,(x + 2 y)}{1 + \left(x y + y^2\right)^2} + z^2 = \dfrac{\partial f}{\partial y}$ ✓

```
In[13]:= CC = D[f, {z, 1}]
```

Out[13]= $2 y z = \dfrac{\partial f}{\partial z}$ ✓

```
In[14]:= {AA, BB, CC} /. {x → 2, y -> 1, z -> -1}
```

Out[14]= {1, 5, -2}

$\operatorname{grad} f(\vec{a}) = (1, 5, -2)$ ✓

```
In[17]:= {1, 5, -2}.{2, -1, -2}
```

Out[17]= 1

$\langle \operatorname{grad} f(\vec{a}) \,|\, \vec{s} \rangle = \langle (1, 5, -2) \,|\, (2, -1, -2) \rangle =$

$= 2 - 5 + 4 = 1$

```
In[18]:= Norm[{2, -1, -2}]
```

Out[18]= 3

$\|\vec{s}\| = \sqrt{4+1+4} = 3$ ✓

$$\frac{\partial f}{\partial \vec{s}}(\vec{a}) = \frac{1}{\|\vec{s}\|} \langle \operatorname{grad} f(\vec{a}) \,|\, \vec{s} \rangle = \frac{1}{3}$$ ✓