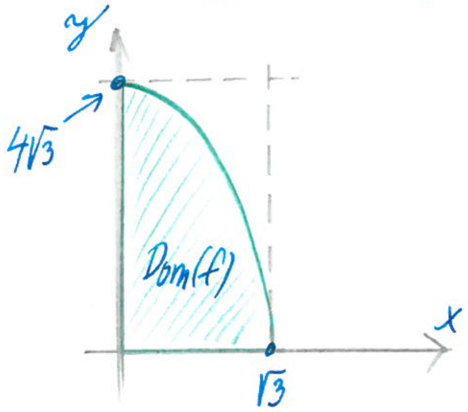
$$f(x,y) = xy\,(48-16x^2-y^2)^{1/2}$$

9 bodů

$$\mathrm{Dom}(f) = \{(x,y)\in\mathbb{R}^2 :\ x,y \geqq 0 \ \wedge\ \underbrace{16x^2+y^2 \leqq 48}_{x^2+\left(\frac{y}{4}\right)^2 \leqq 3}\}$$

elipsa s vrcholy $[\sqrt{3};0]$; $[-\sqrt{3};0]$ $[0;4\sqrt{3}]$; $[0;-4\sqrt{3}]$



Funkční hodnoty na hranici definičního oboru jsou všude nulové, tj. označme-li $E = \mathrm{bd}(\mathrm{Dom}\, f)$, pak $f(E)=0$

$\forall (x,y) \in \mathrm{Dom}(f):\ f(x,y) \geqq 0 \Rightarrow$ všechny body E jsou body lokálních minim

• Stacionární body:

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= y\,(48-16x^2-y^2)^{1/2} - 16yx^2(48-16x^2-y^2)^{-1/2} \overset{!}{=} 0 \\ \frac{\partial f}{\partial y} &= x\,(48-16x^2-y^2)^{1/2} - xy^2(48-16x^2-y^2)^{-1/2} \overset{!}{=} 0 \end{aligned}\right\}$$

body, kde $x=0$ nebo $y=0$, není třeba uvažovat

$$\left.\begin{aligned} 48-16x^2-y^2-16x^2 &= 0 \\ 48-16x^2-y^2-y^2 &= 0 \end{aligned}\right\} \quad y = \pm 4x \quad \Rightarrow \quad \vec{a} = (1,4)$$

• Postačující podmínka:

$$\frac{\partial^2 f}{\partial x^2} = -16xy\,(48-16x^2-y^2)^{-1/2} - 32xy\,(48-16x^2-y^2)^{-1/2} - 16x^2y\,\frac{1}{2}\cdot 16\cdot 2x\,(48-16x^2-y^2)^{-3/2}$$

$$\frac{\partial^2 f}{\partial y^2} = -xy(\ldots\ldots)^{-1/2} - 2xy(\ldots\ldots)^{-1/2} - xy^2\,\frac{1}{2}\cdot 2y\,(\ldots\ldots)^{-3/2}$$

$$\frac{\partial^2 f}{\partial x\partial y} = (\ldots\ldots)^{1/2} - y^2(\ldots)^{-1/2} - 16x^2(\ldots\ldots)^{-1/2} - 16y^2x^2(\ldots\ldots)^{-3/2}$$

$$\frac{\partial^2 f}{\partial x^2}(\vec{a}) = -64 \quad \& \quad \frac{\partial^2 f}{\partial y^2}(\vec{a}) = -4 \quad \& \quad \frac{\partial^2 f}{\partial x\partial y}(\vec{a}) = -8$$

$$d^2f_{\vec{a}}(dx,dy) = -64\,dx^2 - 4\,dy^2 - 16\,dx\,dy = -4\left[16\,dx^2 + dy^2 + 4\,dx\,dy\right]$$

$$= -4\,(dx,dy)\begin{pmatrix} 16 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix} \qquad \Delta_1 = 16 > 0, \quad \Delta_2 = 16-4 > 0$$

$$\Rightarrow \quad d^2f_{\vec{a}}(dx,dy) \lhd 0 \ \ v\ \mathbb{R}^2$$

negativně definitní

v $\vec{a}$ je ostré lokální maximum

8 bodů

$$g(x,y) = \frac{1}{2}xy\underbrace{\left(1 - x^2 - \left(\frac{y}{2}\right)^2\right)^{-1/2}}_{h(x,y)}$$

$h(x,y)$ … začneme hledáním řady pro tuto funkci

$$h(t) = (1+t)^{-1/2} \quad \& \quad t = -x^2 - \left(\frac{y}{2}\right)^2$$

$$h(t) = 1 + \sum_{n=1}^{\infty} \binom{-1/2}{n} t^n = 1 + \sum_{n=1}^{\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\dots\left(-\frac{1}{2}-n+1\right)}{n!} t^n =$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot \dots (2n-1)}{2^n\cdot n!}(-1)^n t^n = 1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!}(-1)^n t^n$$

$$h(x,y) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!}(-1)^n \left(x^2 + \left(\frac{y}{2}\right)^2\right)^n =$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \sum_{k=0}^{n} \binom{n}{k} \frac{y^{2k}}{4^k} x^{2(n-k)} =$$

$$= 1 + \sum_{n=1}^{\infty}\sum_{k=0}^{n} \frac{(2n-1)!!}{(2n)!!}\binom{n}{k}\frac{1}{4^k} y^{2k} x^{2(n-k)}$$

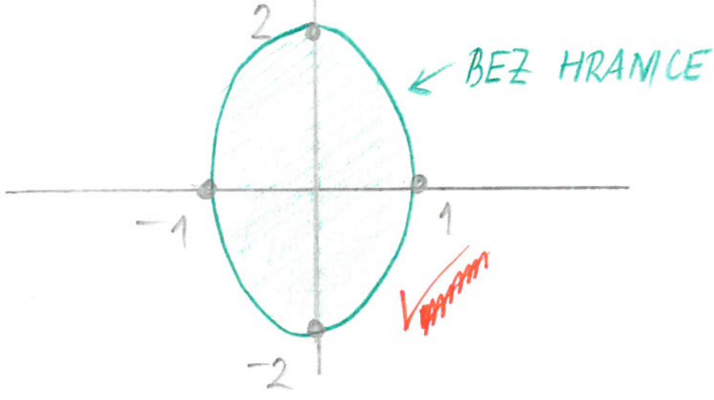
$$g(x,y) = \frac{xy}{2} + \frac{1}{2}\sum_{n=1}^{\infty}\sum_{k=0}^{n} \frac{(2n-1)!!}{(2n)!!}\binom{n}{k}\frac{1}{4^k} y^{2k+1} x^{2n-2k+1}$$

Obor konvergence: Pro $h(t)$: $R^{-1} = \lim_{n\to\infty} \frac{(2n+1)!!}{(2+2n)!!}\frac{(2n)!!}{(2n-1)!!} = 1$ $R=1$

$$\lim_{n\to\infty} n\left(1 - \frac{2n+1}{2n+2}\right) = \lim_{n\to\infty} n\,\frac{2n+2-2n-1}{2n+2} = \frac{1}{2} \Rightarrow \quad O = (-1; 1\rangle$$

$$-1 < -x^2 - \left(\frac{y}{2}\right)^2 \leq 1$$

$$x^2 + \left(\frac{y}{2}\right)^2 < 1$$



$$O = \left\{(x,y)\in\mathbb{R}^2 : x^2 + \left(\frac{y}{2}\right)^2 < 1\right\}$$

9 bodů

$$G(x,y,z) = x^2 - 6xy + 8xz - 4x + 10y^2 - 20yz + 8y + 19z^2 - 20z + 5$$

$$\vec{\lambda} = (-2,0,1) \qquad \& \qquad G(\vec{\lambda}) = 4 - 16 + 8 + 19 - 20 + 5 = 0$$

$$\Rightarrow \vec{\lambda} \text{ je generující bod}$$

Pozor! Nehledá se rozvoj $G(x,y,z)$, ale $z = z(x,y)$!

$$\frac{\partial G}{\partial x} = 2x - 6y + 8z - 4 \qquad \frac{\partial G}{\partial x}(\vec{\lambda}) = -4 + 8 - 4 = 0$$

$$\frac{\partial G}{\partial y} = -6x + 20y - 20z + 8 \qquad \frac{\partial G}{\partial y}(\vec{\lambda}) = 12 - 20 + 8 = 0$$

$$\frac{\partial G}{\partial z} = 8x - 20y + 38z - 20 \qquad \frac{\partial G}{\partial z}(\vec{\lambda}) = -16 + 38 - 20 = 2$$

Generující bod není kritický!

$\Rightarrow$ $(-2,0)$ je stacionární bod implicitní funkce $z = z(x,y)$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{\frac{\partial G}{\partial x}}{\frac{\partial G}{\partial z}}\right) = -\frac{\partial}{\partial x}\left(\frac{2x-6y+8z-4}{8x-20y+38z-20}\right) = \|\text{ve stacionárním bodě lze derivaci podílu zjednodušit}\| =$$

$$= -\frac{2 + 8\frac{\partial z}{\partial x}}{8x-20y+38z-20} \quad \Rightarrow \quad \frac{\partial^2 z}{\partial x^2}(-2,0) = -\frac{2}{2} = -1$$

$$\frac{\partial^2 z}{\partial y^2} = -\frac{20 - 20\frac{\partial z}{\partial y}}{8x-20y+38z-20} \quad \Rightarrow \quad \frac{\partial^2 z}{\partial y^2}(-2,0) = -\frac{20}{2} = -10$$

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{-6 + 8\frac{\partial z}{\partial y}}{8x-20y+38z-20} \quad \Rightarrow \quad \frac{\partial^2 z}{\partial x \partial y}(-2,0) = \frac{6}{2} = 3$$

Taylorův polynom druhého řádu funkce $z(x,y)$ v bodě $(x_0,y_0) = (-2,0)$

$$z = 1 + \overbrace{dz_{(-2,0)}(dx,dy)}^{=0} + \frac{1}{2!} d^2z_{(-2,0)}(dx,dy)$$

$$z = 1 + \frac{1}{2!}\left[-dx^2 - 10\,dy^2 + 6\,dx\,dy\right]$$

$$\underbrace{z = 1 - \frac{1}{2}(x+2)^2 - 5y^2 + 3(x+2)\cdot y}_{\text{Rovnice tečného paraboloidu k zadané ploše v bodě } \vec{\lambda} = (-2,0,1)}$$

* Kdo dělat McL-rozvoj $G(x,y,z)$ je úplně mimo mísu!

$x = r\cos\varphi$
$y = r\sin\varphi$

$$J = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r \overset{!}{\neq} 0$$

Pozor: $M \overset{!}{=} M^0$

$$M_{reg} = \{(r,\varphi) \in \mathbb{R}^2 : r > 0 \wedge \varphi \in (0; 2\pi)\}$$

$$\left.\begin{aligned} 1 &= \frac{\partial r}{\partial x}\cos\varphi - r\frac{\partial\varphi}{\partial x}\sin\varphi \\ 0 &= \frac{\partial r}{\partial x}\sin\varphi + r\frac{\partial\varphi}{\partial x}\cos\varphi \end{aligned}\right\} \Rightarrow \frac{\partial r}{\partial x} = \cos\varphi \;\&\; \frac{\partial\varphi}{\partial x} = -\frac{\sin\varphi}{r}$$

$$\left.\begin{aligned} 0 &= \frac{\partial r}{\partial y}\cos\varphi - r\frac{\partial\varphi}{\partial y}\sin\varphi \\ 1 &= \frac{\partial r}{\partial y}\sin\varphi + r\frac{\partial\varphi}{\partial y}\cos\varphi \end{aligned}\right\} \Rightarrow \frac{\partial r}{\partial y} = \sin\varphi \;\&\; \frac{\partial\varphi}{\partial y} = \frac{\cos\varphi}{r}$$

Odtud:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\cos\varphi - \frac{\partial f}{\partial\varphi}\frac{\sin\varphi}{r} \;\&\; \frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\sin\varphi + \frac{\partial f}{\partial\varphi}\frac{\cos\varphi}{r}$$

Druhé derivace:

$$\frac{\partial^2 f}{\partial x^2} = \cos^2\varphi\frac{\partial^2 f}{\partial r^2} - \frac{\sin(2\varphi)}{r}\frac{\partial^2 f}{\partial\varphi\partial r} + \frac{\sin^2\varphi}{r^2}\frac{\partial^2 f}{\partial\varphi^2} + \frac{\sin^2\varphi}{r}\frac{\partial f}{\partial r} + \frac{\sin(2\varphi)}{r^2}\frac{\partial f}{\partial\varphi}$$

$$\frac{\partial^2 f}{\partial y^2} = \sin^2\varphi\frac{\partial^2 f}{\partial r^2} + \frac{\sin(2\varphi)}{r}\frac{\partial^2 f}{\partial\varphi\partial r} + \frac{\cos^2\varphi}{r^2}\frac{\partial^2 f}{\partial\varphi^2} + \frac{\cos^2\varphi}{r}\frac{\partial f}{\partial r} - \frac{\sin(2\varphi)}{r^2}\frac{\partial f}{\partial\varphi}$$

Dosazení:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\varphi^2} + \frac{1}{r}\frac{\partial f}{\partial r} = \frac{r^2\cos\varphi\sin\varphi}{r^2}$$

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\varphi^2} + \frac{1}{r}\frac{\partial f}{\partial r} = \cos\varphi\cdot\sin\varphi$$

✓... značí půl bod

Hledáme generující bod $\vec{x} = (x_0, y_0, z_0, u_0) = (1, 1, z_0, u_0) = ?$ 8 bodů

$$\left.\begin{array}{ll} -u+1+1+z-2=0 & -u+z=0 \\ u+1+1+z-4=0 & u+z-2=0 \end{array}\right\} \Rightarrow z_0 = 1 \ \& \ u_0 = 1 \checkmark$$

$$\vec{x} = (1,1,1,1)$$

$z = z(x,y)$ & $u = u(x,y)$

• Není náhodou kritický?

$$J = \det\left(\frac{D(F,G)}{D(u,z)}\right) = \begin{vmatrix} -1 & y \\ 1 & y \end{vmatrix} \qquad J(\vec{x}) = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = \underbrace{-2 \neq 0}_{\text{není tedy kritický}} \checkmark$$

• Potřebujeme gradienty funkcí $u = u(x,y)$ & $z = z(x,y)$

$$-\frac{\partial u}{\partial x} + y + y\frac{\partial z}{\partial x} = 0 \qquad -\frac{\partial u}{\partial x}(\vec{a}) + \frac{\partial z}{\partial x}(\vec{a}) + 1 = 0$$

$$\frac{\partial u}{\partial x} + 2xy + y\frac{\partial z}{\partial x} = 0 \qquad \frac{\partial u}{\partial x}(\vec{a}) + \frac{\partial z}{\partial x}(\vec{a}) + 2 = 0$$

$$\frac{\partial z}{\partial x}(\vec{a}) = -\frac{3}{2} \checkmark \qquad \frac{\partial u}{\partial x}(\vec{a}) = -\frac{1}{2} \checkmark$$

$$-\frac{\partial u}{\partial y} + x + 2y + z + y\frac{\partial z}{\partial y} - 2 = 0 \qquad -\frac{\partial u}{\partial y}(\vec{a}) + \frac{\partial z}{\partial y}(\vec{a}) + 2 = 0$$

$$\frac{\partial u}{\partial y} + x^2 + 3y^2 + z + y\frac{\partial z}{\partial y} - 4 = 0 \qquad \frac{\partial u}{\partial y}(\vec{a}) + \frac{\partial z}{\partial y}(\vec{a}) + 1 = 0$$

$$\frac{\partial z}{\partial y}(\vec{a}) = -\frac{3}{2} \checkmark \qquad \frac{\partial u}{\partial y}(\vec{a}) = \frac{1}{2} \checkmark$$

$$\operatorname{grad} z(\vec{a}) = \left(-\frac{3}{2}; \frac{3}{2}\right) \quad \& \quad \operatorname{grad} u(\vec{a}) = \left(-\frac{1}{2}; \frac{1}{2}\right)$$

• Směrové derivace:

$$\frac{\partial z}{\partial \vec{s}}(\vec{a}) = \frac{1}{\sqrt{5}}\left\langle -\frac{3}{2}; -\frac{3}{2} \,\middle|\, (-2,1)\right\rangle = \frac{3}{2}\frac{1}{\sqrt{5}}\langle(-1,1)|(-2,1)\rangle = \frac{3}{2\sqrt{5}} \checkmark$$

$$\frac{\partial u}{\partial \vec{s}}(\vec{a}) = \frac{1}{\sqrt{5}}\left(-\frac{1}{2}; \frac{1}{2}\right)\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \frac{1}{2\sqrt{5}}(-1,1)\cdot\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \frac{3}{2\sqrt{5}} \checkmark$$

> stoupají stejně

$$\frac{\partial z}{\partial \vec{s}}(\vec{a}) = \frac{\partial u}{\partial \vec{s}}(\vec{a}) \Rightarrow$$

~~z(x,y) stoupá strměji~~

obě stoupají stejně strmě ✓

Katedra matematiky Fakulty jaderné a fyzikálně inženýrské ČVUT v Praze							CELKEM
Jméno a příjmení	Cvičící	1	2	3	4	5	

Zápočtová písemná práce č. 1 z předmětu 01ANB4/01MAB4 – verze N

27. dubna 2023, 16:00–18:00

1 (8 bodů)
Nalezněte Maclaurinovu řadu funkce

$$h(x,y) = \frac{1}{\sqrt{1 + 4xy - 4x^2 - y^2}}$$

a poté do obrázku pečlivě načrtněte příslušný obor konvergence.

2 (9 bodů)
Vyšetřete lokální extrémy funkce $f(x,y,z) = xy(x^2 + y^2 - 4) + 2ze^{1-\frac{z}{2}}$ v množině

$$M = \{(x,y,z) \in \mathbf{R}^3 : x > 0 \wedge z > 0\}.$$

3 (7 bodů)
Nalezněte všechny body na ploše

$$y^2 + 4yz - 2x - 10y - 4z + 5z^2 = 3,$$

v nichž je tečná rovina rovnoběžná s rovinou $2x + 6y - 2z = 1$.

4 (9 bodů)
Do parciální diferenciální rovnice

$$3x^2\frac{\partial^2 f}{\partial x^2} - 2xy\frac{\partial^2 f}{\partial x \partial y} - y^2\frac{\partial^2 f}{\partial y^2} + 2y\frac{\partial f}{\partial y} = 0$$

zaveďte nové proměnné $u = \frac{y^3}{x}$ a $v = xy$. Nalezněte také příslušnou maximální množinu regularity.

5 (7 bodů)
Nalezněte obecnou rovnici tečné roviny ke kubické ploše

$$\frac{x^3}{a^3} + \frac{y^3}{b^3} - \frac{z^3}{c^3} = 1$$

v jejím bodě (x_0, y_0, z_0). Upravte do elegantního tvaru. Diskutujte rozsah platnosti výsledku.

$$g(x,y) = \frac{1}{\sqrt{1-(2x-y)^2}}$$

$$h(t) = \frac{1}{\sqrt{1+t}} = (1+t)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} t^n = 1 + \sum_{n=1}^{\infty} \frac{(-\frac{1}{2})(-\frac{3}{2})\dots(-\frac{1}{2}-n+1)}{n!} t^n =$$

$$= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!} t^n \qquad R = 1 \quad \& \quad O = (-1,1)$$

$$g(x,y) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!} (-1)^n (2x-y)^{2n} =$$

$$= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!} \sum_{k=0}^{2n} \binom{2n}{k} 2^k x^k (-1)^{2n-k} \cdot y^{2n-k} =$$

$$= 1 + \sum_{n=1}^{\infty} \sum_{k=0}^{2n} \frac{(2n-1)!!}{(2n)!!} \frac{(2n)!}{k!\,(2n-k)!} (-2)^k x^k y^{2n-k} =$$

$$= 1 + \sum_{n=1}^{\infty} \sum_{k=0}^{2n} (-2)^k \frac{[(2n-1)!!]^2}{k!\,(2n-k)!} x^k y^{2n-k}$$

Obor konvergence: $(2x-y)^2 < 1$

$|2x-y| < 1$

$y < 2x$: $\quad 2x - y < 1$, $\quad y > 1+2x$

$y \geq 2x$: $\quad -2x + y < 1$, $\quad y < 1+2x$

& obrázek

9b

In[31]:= `H := x y (x^2 + y^2 - 4) + 2 z Exp[1 - z / 2]`

In[32]:= `Solve[Reduce[{∂x H == 0, ∂y H == 0, ∂z H == 0, x > 0, z > 0}, {x, y, z}], {x, y, z}]`

Out[32]= $\{\{x \to 1, y \to -1, z \to 2\}, \{x \to 1, y \to 1, z \to 2\}, \{x \to 2, y \to 0, z \to 2\}\}$

In[33]:= `sol1 = {x → 1, y → -1, z → 2}`

Out[33]= $\{x \to 1, y \to -1, z \to 2\}$

In[34]:= `d2H = (∂x,x H dx^2 + ∂y,y H dy^2 + ∂z,z H dz^2 + 2 ∂x,y H dx dy + 2 ∂x,z H dx dz + 2 ∂y,z H dy dz) /. sol1`

Out[34]= $-6\,dx^2 + 4\,dx\,dy - 6\,dy^2 - dz^2$

$\begin{pmatrix} -6 & 2 & 0 \\ 2 & -6 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $\left.\begin{matrix} 36-4>0 \\ -6<0 \end{matrix}\right\}$ OL max.

In[35]:= `sol2 = {x → 1, y → 1, z → 2}`

Out[35]= $\{x \to 1, y \to 1, z \to 2\}$

In[36]:= `d2H = (∂x,x H dx^2 + ∂y,y H dy^2 + ∂z,z H dz^2 + 2 ∂x,y H dx dy + 2 ∂x,z H dx dz + 2 ∂y,z H dy dz) /. sol2`

Out[36]= $6\,dx^2 + 4\,dx\,dy + 6\,dy^2 - dz^2$ NENÍ EXTRÉM

In[37]:= `sol3 = {x → 2, y → 0, z → 2}`

Out[37]= $\{x \to 2, y \to 0, z \to 2\}$

In[38]:= `d2H = (∂x,x H dx^2 + ∂y,y H dy^2 + ∂z,z H dz^2 + 2 ∂x,y H dx dy + 2 ∂x,z H dx dz + 2 ∂y,z H dy dz) /. sol3`

Out[38]= $16\,dx\,dy - dz^2$ NENÍ EXTRÉM

$$\left.\begin{aligned} y(x^2+y^2-4) + 2x^2y &= 0 \\ x(x^2+y^2-4) + 2xy^2 &= 0 \\ 2e^{1-z/2} - z e^{1-z/2} &= 0 \end{aligned}\right\} \checkmark \quad \Rightarrow \quad \begin{aligned} \vec{a} &= (1,-1,2) \checkmark \\ \vec{b} &= (1,1,2) \checkmark \\ \vec{c} &= (2,0,2) \checkmark \end{aligned}$$

$$\checkmark\left\{\begin{aligned} &\frac{\partial^2 h}{\partial x^2} = 2yx + 4xy = 6yx \qquad \frac{\partial^2 h}{\partial y^2} = 2xy + 4xy \qquad \frac{\partial^2 h}{\partial x \partial y} = x^2+y^2-4+2x^2 \quad + 2y^2 \\ &\frac{\partial^2 h}{\partial z^2} = -\cancel{2}e^{1-z/2} + \frac{z}{2} e^{1-z/2} \qquad \frac{\partial^2 h}{\partial x \partial z} = \frac{\partial^2 h}{\partial y \partial z} = 0 \end{aligned}\right.$$

$\begin{pmatrix} -6 & 2 & 0 \\ 2 & -6 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Sylvester: $-6<0$, $36-4>0$, $-32<0$

$$\checkmark\left\{\begin{aligned} d^2H_{\vec{a}} &= -6dx^2 + 4dxdy - 6dy^2 - dz^2 \quad \lhd 0 \\ d^2H_{\vec{b}} &= 6dx^2 + 4dxdy + 6dy^2 - dz^2 \quad \lhd\rhd 0 \\ d^2H_{\vec{c}} &= 16dxdy - dz^2 \quad \lhd\rhd 0 \end{aligned}\right.$$

určení typu definitnosti ✓

$$\left.\begin{aligned} &\vec{a} \ldots \text{ OL maximum} \\ &\vec{b}, \vec{c} \ldots \text{ sedlové body} \end{aligned}\right\} \checkmark$$

Nalezněte takovou tečnou rovinu k ploše

$$y^2 + 5z^2 + 4yz - 2x - 10y - 4z = 3.$$

která je rovnoběžná s rovinou $3x + 9y - 3z = 1$. V zadání: $2x + 6y - 2z = 1$

$F(x,y,z) = y^2 + 5z^2 + 4yz - 2x - 10y - 4z - 3 = 0$ zadává např. $z = z(x,y)$

Existence: $\frac{\partial F}{\partial z} = 10z + 4y - 4 \neq 0$ (nebo níže ※)

Tečná rovina ke grafu funkce $z = z(x,y)$: $z - z_0 = \frac{\partial z}{\partial x}(x_0,y_0)\,dx + \frac{\partial z}{\partial y}(x_0,y_0)\,dy$

$$\frac{\partial z}{\partial x}(x_0,y_0)\cdot x + \frac{\partial z}{\partial y}(x_0,y_0)y - 1 = \frac{\partial z}{\partial x}(x_0,y_0)x_0 + \frac{\partial z}{\partial y}(x_0,y_0)y_0 + z_0$$

normálový vektor: $\vec{n} = \left(\frac{\partial z}{\partial x}(x_0,y_0); \frac{\partial z}{\partial y}(x_0,y_0); -1\right)$

normálový vektor ze zadání:

$\vec{u} = (2, 6, -2)$

potřebujeme, aby $v_3 = -1 \Rightarrow \vec{v} = (1, 3, -1)$

$\Downarrow$

$\frac{\partial z}{\partial x}(x_0,y_0) = 1 \wedge \frac{\partial z}{\partial y}(x_0,y_0) = 3$

$10z\frac{\partial z}{\partial x} - 2 - 4\frac{\partial z}{\partial x} + 4y\frac{\partial z}{\partial x} = 0$ $\qquad \frac{\partial z}{\partial x} = \frac{2}{10z + 4y - 4}$

$2y + 10z\frac{\partial z}{\partial y} + 4z + 4y\frac{\partial z}{\partial y} - 10 - 4\frac{\partial z}{\partial y} = 0$ $\qquad \frac{\partial z}{\partial y} = \frac{10 - 2y - 4z}{10z + 4y - 4}$

$\frac{2}{10z_0 + 4y_0 - 4} = 1 \wedge \frac{10 - 2y_0 - 4z_0}{10z_0 + 4y_0 - 4} = 3$

$10 - 2y_0 - 4z_0 = 6 \Rightarrow y_0 = 2 - 2z_0$

$\Downarrow$

$10z_0 + 8 - 8z_0 - 4 = 2$

$z_0 = -1$

$16 + 5 - 16 - 2x_0 - 40 + 4 - 3 = 0$ $\qquad y_0 = 4 \Leftarrow$

$2x_0 = -34 \Rightarrow x_0 = -17$

$\vec{x} = (-17, 4, -1)$ $\qquad \frac{\partial F}{\partial z}(\vec{x}) = -10 + 16 - 4 = 2$ (※)

Tečná rovina: $x + 3y - z = -17 + 3\cdot 4 + 1$

$x + 3y - z + 4 = 0$

Parciální diferenciální rovnici

$$3x^2\frac{\partial^2 f}{\partial x^2} - 2xy\frac{\partial^2 f}{\partial x \partial y} - y^2\frac{\partial^2 f}{\partial y^2} + 2y\frac{\partial f}{\partial y} = 0$$

pro neznámou funkci $z(x,y)$ substituujte transformačními vztahy

$$u = \frac{y^3}{x}, \qquad v = xy.$$

Stanovte příslušnou maximální množinu regularity zadaného zobrazení a výsledek transformace upravte do nejjednoduššího možného tvaru.

$$\det\left(\frac{D(u,v)}{D(x,y)}\right) = \begin{vmatrix} -\frac{y^3}{x^2} & 3\frac{y^2}{x} \\ y & x \end{vmatrix} = -\frac{y^3}{x} - 3\frac{y^3}{x} = -4\frac{y^3}{x} \neq 0$$

$$M_{reg} = \{(x,y) \in E^2 : y \neq 0 \wedge x \neq 0\}$$

$$\frac{\partial f}{\partial x} = -\frac{y^3}{x^2}\frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v} \qquad \frac{\partial f}{\partial y} = 3\frac{y^2}{x}\frac{\partial f}{\partial u} + x\frac{\partial f}{\partial v}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{y^6}{x^4}\frac{\partial^2 f}{\partial u^2} - 2\frac{y^4}{x^2}\frac{\partial^2 f}{\partial u \partial v} + y^2\frac{\partial^2 f}{\partial v^2} + 2\frac{y^3}{x^3}\frac{\partial f}{\partial u}$$

$$\frac{\partial^2 f}{\partial y^2} = 9\frac{y^4}{x^2}\frac{\partial^2 f}{\partial u^2} + 6y^2\frac{\partial^2 f}{\partial u \partial v} + x^2\frac{\partial^2 f}{\partial v^2} + 6\frac{y}{x}\frac{\partial f}{\partial u}$$

$$\frac{\partial^2 f}{\partial x \partial y} = -3\frac{y^5}{x^3}\frac{\partial^2 f}{\partial u^2} + \left(3\frac{y^3}{x} - \frac{y^3}{x}\right)\frac{\partial^2 f}{\partial u \partial v} + xy\frac{\partial^2 f}{\partial v^2} - 3\frac{y^2}{x^2}\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$$

Dosazení:

$$\frac{\partial^2 f}{\partial u^2}: \; 3\frac{y^6}{x^2} - 9\frac{y^6}{x^2} + 6\frac{y^6}{x^2} = 0 \qquad \frac{\partial^2 f}{\partial u \partial v}: \; -6y^4 - 6y^4 - 4y^4 = -16y^4$$

$$\frac{\partial^2 f}{\partial v^2}: \; 3x^2y^2 - x^2y^2 - 2x^2y^2 = 0 \qquad \frac{\partial f}{\partial u}: \; 6\frac{y^3}{x} - 6\frac{y^3}{x} + 6\frac{y^3}{x} + 6\frac{y^3}{x} = 12\frac{y^3}{x}$$

$$\frac{\partial f}{\partial v}: \; -2xy + 2xy = 0$$

$$-16y^4\frac{\partial^2 f}{\partial u \partial v} + 12\frac{y^3}{x}\frac{\partial f}{\partial u} = 0$$

$$\frac{\partial^2 f}{\partial u \partial v} - \frac{3}{4}\frac{1}{xy}\frac{\partial f}{\partial u} = 0$$

$$\frac{\partial^2 f}{\partial u \partial v} - \frac{3}{4v}\frac{\partial f}{\partial u} = 0$$

1+6=7

$$\frac{x^3}{a^3} + \frac{y^3}{b^3} - \frac{z^3}{c^3} = 1 \qquad \bar{X} = (x_0; y_0; z_0)$$

$$z = z(x, y) \qquad \frac{\partial F}{\partial z} = -3\frac{z^2}{c^3} \neq 0 \Rightarrow z \neq 0$$

$$\frac{\partial F}{\partial x} = 3\frac{x^2}{a^3} \qquad \frac{\partial F}{\partial y} = 3\frac{y^2}{b^3}$$

$$\bar{a} = (x_0; y_0) \qquad z(\bar{a}) = z_0$$

$$\frac{\partial z}{\partial x}(\bar{a}) = \frac{c^3}{a^3}\frac{x_0^2}{z_0^2} \qquad \frac{\partial z}{\partial y}(\bar{a}) = \frac{c^3}{b^3}\frac{y_0^2}{z_0^2}$$

Tečná rovina:

$$z - z_0 = \frac{c^3}{a^3}\frac{x_0^2}{z_0^2}(x - x_0) + \frac{c^3}{b^3}\frac{y_0^2}{z_0^2}(y - y_0) \quad / \, z_0^2 \frac{1}{c^3}$$

$$\frac{z z_0^2 - z_0^3}{c^3} = \frac{x x_0^2 - x_0^3}{a^3} + \frac{y y_0^2 - y_0^3}{b^3}$$

$$\frac{x x_0^2}{a^3} + \frac{y y_0^2}{b^3} - \frac{z z_0^2}{c^3} = \frac{x_0^3}{a^3} + \frac{y_0^3}{b^3} - \frac{z_0^3}{c^3}$$

$$\frac{x x_0^2}{a^3} + \frac{y y_0^2}{b^3} - \frac{z z_0^2}{c^3} = 1$$

Zdá se, že výsledek platí pouze, pokud $z \neq 0$. Pokud by ale $z = 0$, pak by buď $x \neq 0$ nebo $y \neq 0$ a bylo by možné příklad přes implicitní funkci $x = x(y, z)$ nebo $y = y(x, z)$

$\Downarrow$

výsledek je platný univerzálně!