$$g(x) = \Theta(x)\,\lambda\,e^{-\lambda x} \stackrel{\Delta}{=} h(x)$$

$$g_k(x) = \mathop{*}_{i=0}^{k} h(x) \quad / \ \mathcal{L}$$

$$\mathcal{L}[g_k] = \prod_{i=0}^{k} \mathcal{L}[h(x)] = H^{k+1}(s)$$

$$H(s) = \mathcal{L}[h] = \mathcal{L}[\Theta(x)\,\lambda\,e^{-\lambda x}] = \frac{\lambda}{\lambda+s}$$

$$\mathcal{L}[g_k] = \frac{\lambda^{k+1}}{(\lambda+s)^{k+1}}$$

víme, že $\mathcal{L}[\Theta(x)\,x^k] = \dfrac{k!}{s^{k+1}}$

odtud: $\mathcal{L}[\Theta(x)\,x^k e^{-\lambda x}] = \dfrac{k!}{(s+\lambda)^{k+1}} =$

$$= \frac{k!}{\lambda^{k+1}} \cdot \frac{\lambda^{k+1}}{(s+\lambda)^{k+1}}$$

$$\Rightarrow \quad g_k(x) = \frac{\lambda^{k+1}}{k!}\,x^k e^{-\lambda x} \cdot \Theta(x)$$

$$P[\eta_L = 0] = 1 - \int_0^L g_0(x)\,dx = 1 - \int_0^L \lambda e^{-\lambda x}\,dx = 1 + \left[e^{-\lambda x}\right]_0^L = e^{-\lambda L}$$

$$P[\eta_L = k] = \int_0^L g_{k-1}(x)\,dx - \int_0^L g_k(x)\,dx = \int_0^L \left(\cancel{\frac{(k-1)!^{-1}}{\lambda^k}}\right) \frac{\lambda^k}{(k-1)!}\,x^{k-1} e^{-\lambda x}\,dx$$

$$- \int_0^L \frac{\lambda^{k+1}}{k!}\,x^k e^{-\lambda x}\,dx = \left| \begin{matrix} u = x^k & v' = e^{-\lambda x} \\ u' = k\,x^{k-1} & v = -\frac{1}{\lambda} e^{-\lambda x} \end{matrix} \right| =$$

$$= \int_0^L \frac{\lambda^k}{(k-1)!}\,x^{k-1} e^{-\lambda x}\,dx + \left[\frac{\lambda^{k+1}}{k!}\,x^k \frac{1}{\lambda} e^{-\lambda x}\right]_0^L - \int_0^L \frac{\lambda^{k+1}}{k!}\,x^{k-1} \cdot k \cdot \frac{1}{\lambda} e^{-\lambda x}\,dx =$$

$$= \frac{\lambda^k L^k}{k!}\,e^{-\lambda L}$$

a právě toto je Poissonovo rozdělení

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$$R(s) = \mathcal{L}\Big[\sum_{m=0}^{\infty} g_m(x)\Big] \overset{(*)}{=} \sum_{m=0}^{\infty} \mathcal{L}[g_m(x)] = \sum_{m=0}^{\infty} G_m(s) =$$

$$= \left\| \begin{array}{l} g_m(x) = \mathop{\ast}\limits_{i=0}^{m} h(x) \\ G_m(s) = H^{m+1}(s) \leftarrow \text{konvoluční teorém} \end{array} \right\| \overset{(**)}{=} \sum_{m=0}^{\infty} H^{m+1}(s) =$$

$$= \Big\| \, |H(s)| < 1 \text{ na } (0, +\infty) \, \Big\| = \frac{H(s)}{1 - H(s)}$$

- $H(s)$ klesá na $\langle 0; +\infty)$ $\Leftarrow$ $H'(s) = -\int_0^{+\infty} x \cdot h(x) e^{-sx} \, dx < 0$
- $H(0) = 1$ $\Leftarrow$ $H(0) = \int_0^{+\infty} h(x) \cdot e^{0 \cdot x} \, dx = \|h\| = 1$
- $H(s) \neq 1$ na $(0; +\infty)$ … sporem

$$\int_0^{\infty} h(x) e^{-sx} \, dx = 1 = \int_0^{+\infty} h(x) \, dx$$

$$\Rightarrow \int_0^{\infty} h(x) \cdot [1 - e^{-sx}] \, dx = 0 \Rightarrow$$

$$\Rightarrow 1 - e^{-sx} \sim 0 \quad \text{spor}$$

(*) σ-aditivita Laplaceovy transformace v $\mathcal{B}$

(**) součet geometrické řady

$$y = x_1 + x_2 + x_3 \qquad X_k \sim h(x) \qquad \Rightarrow \qquad g(y) = (h * h * h)(y)$$

$$g(y) \sim y$$

$$(h * h)(x) = \int_{\mathbb{R}} \theta(y) \sqrt{y}\, e^{-\frac{\alpha}{y}} e^{-\lambda y}\, \theta(x-y) \sqrt{x-y}\, e^{-\frac{\alpha}{x-y}} e^{-\lambda(x-y)}\, dy =$$

$$= \theta(x)\, e^{-\lambda x} \int_0^x \sqrt{y} \sqrt{x-y}\, e^{-\left[\frac{\alpha}{y} + \frac{\alpha}{x-y}\right]} dy =$$

$$= \left\| \begin{array}{ll} g(y) \triangleq \frac{\alpha}{y} + \frac{\alpha}{x-y} \quad \text{na } y \in (0; x) & g\left(\frac{x}{2}\right) = \frac{2\alpha}{x} + \frac{2\alpha}{x} = \frac{4\alpha}{x} \\ g'(y) = -\frac{\alpha}{y^2} + \frac{\alpha}{(x-y)^2} \stackrel{!}{=} 0 \quad \Rightarrow \quad y = \frac{x}{2} & \end{array} \right\| =$$

$$\doteq \left\| \begin{array}{l} \text{aproximace metodou} \\ \text{hrubého leadingu} \end{array} \right\| \doteq \theta(x)\, e^{-\lambda x} e^{-\frac{4\alpha}{x}} \int_0^x \sqrt{y} \sqrt{x-y}\, dy =$$

$$= \theta(x)\, e^{-\lambda x} e^{-\frac{\alpha \cdot 4}{x}} \cdot \frac{\pi x^2}{8}$$

$$(h * h * h)(x) = \int_{\mathbb{R}} \theta(y)\, e^{-\lambda y} e^{-\frac{4\alpha}{y}} \cdot \frac{\pi y^2}{8}\, \theta(x-y) \sqrt{x-y}\, e^{-\frac{\alpha}{x-y}} e^{-\lambda(x-y)}\, dy =$$

$$= \theta(x)\, e^{-\lambda x} \frac{\pi}{8} \int_0^x y^2 \sqrt{x-y} \cdot e^{-\left[\frac{4\alpha}{y} + \frac{\alpha}{x-y}\right]} dy =$$

$$= \left\| \begin{array}{lll} h(y) = \frac{4\alpha}{y} + \frac{\alpha}{x-y} & 2(x-y) = y & h\left(\frac{2}{3}x\right) = \frac{4\alpha}{2x} \cdot 3 + \frac{\alpha}{x} \cdot 3 = \\ \frac{1}{\alpha} h'(y) = -\frac{4}{y^2} + \frac{1}{(x-y)^2} \stackrel{!}{=} 0 & y = \frac{2}{3} x & = \frac{\alpha}{x}(6+3) = \frac{9\alpha}{x} \end{array} \right\| =$$

$$= \theta(x)\, e^{-\lambda x} \frac{\pi}{8} e^{-\frac{9\alpha}{x}} \int_0^x y^2 \sqrt{x-y}\, dy = \theta(x)\, e^{-\lambda x} e^{-\frac{9\alpha}{x}} \frac{\pi}{8} \cdot \frac{16\, x^{7/2}}{105} =$$

$$= \theta(x)\, \frac{2\pi}{105}\, x^{7/2}\, e^{-\frac{9\alpha}{x}} e^{-\lambda x}$$

i)

$$g_1(x) = \Theta(x)\int_0^x h(y)\,h(x-y)\,dy \leq K^2\cdot\Theta(x)\cdot x \quad \Leftarrow \quad h(x)\in\mathcal{B} \text{ a } h(x) \text{ je omezená}$$

$$g_2(x) = \Theta(x)\int_0^x g_1(y)\,h(x-y)\,dy \leq \Theta(x)\,K^3\int_0^x y\,dy = \Theta(x)\,\frac{K^3}{2}x^2$$

$$g_3(x) = \Theta(x)\int_0^x g_2(x)\,h(x-y)\,dy \leq \Theta(x)\,\frac{K^4}{2}\int_0^x y^2\,dy = \Theta(x)\,\frac{K^4}{3!}x^3$$

$$g_m(x) \leq K^{m+1}\frac{L^m}{m!}$$

$$m\cdot g_m(x) \leq K^{m+1}\frac{L^m}{(m-1)!} \quad \Big/ \sum_m$$

$$\sum_{m=1}^{\infty} m\cdot g_m(x) \leq \underbrace{\sum_{m=1}^{\infty} K^{m+1}\frac{L^m}{(m-1)!} = \Big|\, m=n+1 \,\Big\| = K^2\cdot L\sum_{n=0}^{\infty}\frac{(KL)^n}{n!} = K^2\cdot L\,e^{KL}}_{\text{číselná majoranta tedy konvergují}}$$

$\Rightarrow$ podle Weierstrasse: $\sum m\,g_m(x) \rightrightarrows$ $\langle 0,a\rangle$

ii)

$$E(\eta_L^2) = \sum_{k=1}^{\infty} k^2 P[\eta_L = k] = \sum_{k=1}^{\infty} k^2\left(\int_0^L (g_{k-1}(x) - g_k(x))\,dx\right) = \left\|\,\underline{\text{řada SK na } \langle 0,L\rangle}\,\right\| =$$

$$= \int_0^L \sum_{k=1}^{\infty} k^2(g_{k-1}(x) - g_k(x))\,dx = \left\|\,\text{obě řady konvergují}\,\right\| =$$

$$= \int_0^L \sum_{k=1}^{\infty} k^2 g_{k-1}(x)\,dx - \int_0^L \sum_{k=2}^{\infty}(k-1)^2 g_{k-1}(x)\,dx =$$

$$= \int_0^L \sum_{k=1}^{\infty}(k^2 - k^2 + 2k - 1)\,g_{k-1}(x)\,dx = \int_0^L \sum_{k=1}^{\infty}(2k-1)\,g_{k-1}(x)\,dx =$$

$$= \int_0^L \sum_{m=0}^{\infty}(2m+1)\,g_m(x)\,dx = 2\int_0^L \sum_{m=0}^{\infty} m\cdot g_m(x)\,dx + \int_0^L \underbrace{\sum_{m=0}^{\infty} g_m(x)}_{r(x)}\,dx =$$

$$= \left\|\,\text{z nápovědy dále plyne}\,\right\| = \underbrace{2\int_0^L (r * r)(x)\,dx + \int_0^L r(x)\,dx}$$

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$$s \cdot H(s) = s \cdot \mathcal{L}\left[\Theta(x) \int_x^{+\infty} g(y)\, dy\right] = s \cdot \mathcal{L}\left[\Theta(x) \int_{\mathbb{R}} \Theta(y-x)\, g(y)\, dy\right] =$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \Theta(x)\, \Theta(y-x)\, g(y)\, s\, e^{-sx}\, d(x,y) \overset{F.V.}{=}$$

$$= \int_{\mathbb{R}} g(y) \cdot \left\{ \int_{\mathbb{R}} \Theta(x)\Theta(y-x) \cdot s \cdot e^{-sx} dx \right\} dy =$$

$$= \int_{\mathbb{R}} g(y) \cdot \left\{ \int_0^y s \cdot e^{-sx} dx \right\} dy = \int_{\mathbb{R}} g(y) \cdot \left[-e^{-sx} \right]_0^y dy =$$

$$= \int_{\mathbb{R}} g(y) \cdot \left[1 - e^{-sy} \right] dy = \int_{\mathbb{R}} g(y)\, dy - \int_{\mathbb{R}} g(y) e^{-sy} dy =$$

$$= \|g\| - G(s)$$

$$\Rightarrow \quad \mathcal{L}\left[\Theta(x) \int_x^{+\infty} g(y)\, dy\right] = \frac{\mathcal{M}_0 - G(s)}{s}$$

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$$\mathcal{L}\left[\int_x^{+\infty} g(y)\, dy\right] = \frac{\mathcal{M}_0 - G(s)}{s} = \frac{\mathcal{M}_0 - \sum_{i=0}^{\infty} (-1)^i \frac{\mathcal{M}_i}{i!} s^i}{s} =$$

$$= \sum_{i=1}^{\infty} (-1)^{i+1} \frac{\mathcal{M}_i}{i!} s^{i-1} = \left\| \begin{matrix} n = i-1 \\ i = n+1 \end{matrix} \right\| =$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{\mathcal{M}_{n+1}}{(n+1)!} s^n \overset{!}{=} \sum_{n=0}^{\infty} (-1)^n \frac{\lambda_n}{n!} s^n$$

$$\Rightarrow \quad \frac{\mathcal{M}_{n+1}}{(n+1)!} \overset{!}{=} \frac{\lambda_n}{n!} \quad \Rightarrow \quad \lambda_n = \frac{\mathcal{M}_{n+1}}{n+1}$$

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